

# WONDERFUL COMPACTIFICATION OF CHARACTER VARIETIES

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**ABSTRACT.** Using the wonderful compactification of a semisimple adjoint affine algebraic group  $G$  defined over an algebraically closed field  $\mathbb{k}$  of arbitrary characteristic, we construct a natural compactification  $\overline{\mathfrak{X}}_\Gamma(G)$  of the  $G$ -character variety of any finitely generated group  $\Gamma$ . When  $\Gamma$  is a free group, we show that this compactification is always simply connected with respect to the étale fundamental group, and when  $\mathbb{k} = \mathbb{C}$  it is also topologically simply connected. For other groups  $\Gamma$ , we describe conditions for the compactification of the moduli space to be simply connected and give examples when these conditions are satisfied, including closed surface groups and free abelian groups when  $G = \mathrm{PGL}_n(\mathbb{C})$ .

Additionally, when  $\Gamma$  is free we identify the boundary divisors of  $\overline{\mathfrak{X}}_\Gamma(G)$  in terms of previously studied moduli spaces, and we construct a Poisson structure on  $\overline{\mathfrak{X}}_\Gamma(G)$  and its boundary divisors.

## 1. INTRODUCTION

To understand how groups  $\Gamma$  act on spaces  $X$  one considers homomorphisms  $\Gamma \rightarrow \mathrm{Aut}(X)$ . When  $\mathrm{Aut}(X)$  is an algebraic group  $G$ , the collection of homomorphisms  $\mathrm{Hom}(\Gamma, G)$  is an algebraic variety and so deformation techniques are available. From the associated study of  $G$ -local systems, two homomorphisms are equivalent when they are conjugate via an element of  $G$ . In this case, the quotient space  $\mathrm{Hom}(\Gamma, G)/G$  is naturally considered. Unfortunately this quotient space is not generally algebraic and so deformation techniques are not available. An approximation to this space, that often has better properties, is called the  $G$ -character variety of  $\Gamma$ . It will be denoted by  $\mathfrak{X}_\Gamma(G)$ .

When  $G$  is a reductive algebraic group over an algebraically closed field  $\mathbb{k}$ , the above mentioned space  $\mathfrak{X}_\Gamma(G)$  is precisely the geometric invariant theoretic (GIT) quotient  $\mathrm{Hom}(\Gamma, G)//G$ ; in other words, it is the spectrum of the ring of invariants  $\mathbb{k}[\mathrm{Hom}(\Gamma, G)]^G$ .

Considering families lying in  $\mathfrak{X}_\Gamma(G)$  demands an understanding of (geometrically meaningful) boundary divisors, and as such compactifications of  $\mathfrak{X}_\Gamma(G)$  precipitate.

For example, in [MS84], a compactification of  $\mathrm{SL}_2(\mathbb{C})$ -character varieties by actions on  $\mathbb{R}$ -trees gave a new proof of Thurston's theorem that projective measured geodesic laminations give a compactification of Teichmüller space; the latter gives a classification of surface group automorphisms. More recently, in [Man15], it was shown that each quiver-theoretic avatar of a free group character variety developed in [FL13] determines a natural compactification, under the assumption that  $G$  is simple and simply connected over  $\mathbb{C}$ . And in [Kom15], compactifications of relative character varieties of punctured

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spheres are considered in order to understand the relationship between the Dolbeault moduli space of Higgs bundles and the Betti moduli space of representations.

In this paper, we prove the following theorem.

**Theorem 1.1.** *Let  $G$  be an adjoint type semisimple algebraic group defined over an algebraically closed field  $\mathbb{k}$ . Then the wonderful compactification of  $G$  determines a compactification, denoted by  $\overline{\mathfrak{X}}_\Gamma(G)$ , of  $\mathfrak{X}_\Gamma(G)$  for any finitely generated group  $\Gamma$ . If  $\Gamma$  is a free group, then  $\overline{\mathfrak{X}}_\Gamma(G)$  is étale simply connected. If  $\mathbb{k} = \mathbb{C}$ , then there exists a compactification  $\overline{\mathfrak{X}}_\Gamma(G)$  that is both topologically and étale simply connected whenever  $\mathfrak{X}_\Gamma(G)$  is simply connected and normal.*

This result follows from Theorem 3.1, Corollary 4.2 and Lemma 4.3.

In Proposition 4.5, we give examples when the latter condition of normality holds:

- (1)  $\Gamma$  is a free group,
- (2)  $\Gamma$  is a surface group and  $G = \mathrm{PGL}_n(\mathbb{C})$ , or
- (3)  $\Gamma$  is free abelian and  $G$  does not have exceptional factors.

In Sections 5 and 6 we further study the case in which  $\Gamma$  is a free group. We identify the boundary divisors of  $\overline{\mathfrak{X}}_\Gamma(G)$  (Theorem 5.2) in terms of the *parabolic character varieties* studied by Biswas–Florentino–Lawton–Logares [BFL14], and we construct a Poisson structure on  $\overline{\mathfrak{X}}_\Gamma(G)$  and on its boundary divisors (Theorem 6.4) using work of Evens–Lu [EL01, EL06], who constructed a Poisson structure on  $\overline{G}$ .

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## 2. WONDERFUL COMPACTIFICATION OF GROUPS

Let  $G$  be a connected affine algebraic group defined over an algebraically closed field  $\mathbb{k}$ ; there is no condition on its characteristic. Let  $\mathfrak{g} = \mathrm{Der}_{\mathbb{k}}(\mathbb{k}[G], \mathbb{k})^G$  be the Lie algebra of  $G$ , where  $G$  acts on the derivations via the left-translation action of  $G$  on itself. The group  $G$  is said to be of *adjoint type* if the adjoint representation

$$\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \tag{2.1}$$

is an embedding. The center of an adjoint type group is trivial.

We will always assume that  $G$  is semisimple of adjoint type. Therefore,  $G$  is of the form  $\prod_{i=1}^m (G_i/Z_i)$ , where each  $G_i$  is a simple simply connected group and  $Z_i$  is the center of  $G_i$ .

In [DCP83], assuming the base field is of characteristic 0, a compactification of  $G$  is constructed, called the *wonderful compactification*. In [Str87] the construction is generalized to arbitrary characteristic. Denote the wonderful compactification of  $G$  by  $\overline{G}$ . In [EL01, EL06], a Poisson structure on  $\overline{G}$  is constructed when the characteristic of the base field is zero.

We now describe the construction of  $\overline{G}$ , following the exposition in [EL01, EJ08]. Let  $n$  be the dimension of  $G$ . The general linear group  $\mathrm{GL}(\mathfrak{g} \oplus \mathfrak{g})$  acts on the space of  $n$ -dimensional subspaces of  $\mathfrak{g} \oplus \mathfrak{g}$  transitively with the stabilizer of a point being a parabolic subgroup  $P$ . The Grassmannian  $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) = \mathrm{GL}(\mathfrak{g} \oplus \mathfrak{g})/P$  of dimension  $(2n)^2 - 3n^2 = n^2$  parametrizes the  $n$ -dimensional subspaces of  $\mathfrak{g} \oplus \mathfrak{g}$ . Consider the composition homomorphism

$$G \times G \xrightarrow{\rho \times \rho} \mathrm{GL}(\mathfrak{g}) \times \mathrm{GL}(\mathfrak{g}) \hookrightarrow \mathrm{GL}(\mathfrak{g} \oplus \mathfrak{g}),$$

where  $\rho$  is the homomorphism in (2.1) and  $\mathrm{GL}(\mathfrak{g}) \times \mathrm{GL}(\mathfrak{g})$  is the subgroup of automorphisms of  $\mathfrak{g} \oplus \mathfrak{g}$  that preserves the decomposition. This homomorphism produces an action of  $G \times G$  on  $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ . Let

$$\mathfrak{g}_\Delta := \{(x, x) \mid x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}$$

be the diagonal subalgebra, which is an  $n$ -dimensional subspace and hence a point in  $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ . The stabilizer of this subspace  $\mathfrak{g}_\Delta$  for the above action of  $G \times G$  on  $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$  is

$$G_\Delta := \{(g, g) \mid g \in G\}.$$

Therefore, the orbit of  $\mathfrak{g}_\Delta$  is

$$(G \times G) \cdot \mathfrak{g}_\Delta = (G \times G)/G_\Delta \cong G.$$

The wonderful compactification of  $G$  is then  $\overline{G} = \overline{(G \times G) \cdot \mathfrak{g}_\Delta}$ , where the closure is taken inside  $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ , making  $\overline{G}$  an irreducible projective variety containing  $G = (G \times G) \cdot \mathfrak{g}_\Delta$  as a Zariski open subvariety.

**Theorem 2.1** ([Str87], [DCP83]). *The following properties hold for the wonderful compactification  $\overline{G}$ :*

- (1) *The action of  $G \times G$  on  $G$ , defined by  $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ , extends to a  $G \times G$  action on  $\overline{G}$ .*
- (2)  *$\overline{G}$  is smooth.*
- (3) *Each  $G \times G$  orbit closure in  $\overline{G}$  is smooth.*
- (4) *The complement  $\overline{G} \setminus G$  consists of  $r = \mathrm{rank}(G)$  smooth divisors  $D_1, \dots, D_r$  with simple normal crossings, each of which is the closure of a single  $G \times G$  orbit.*

Note that the diagonal  $G_\Delta = G$  acts by conjugation on  $\overline{G}$ . We now show that  $\overline{G}$  is simply connected, after reminding the reader of requisite terms.

A morphism of irreducible normal projective varieties  $f : Y \rightarrow X$  is *étale* if the induced map  $\widehat{\mathcal{O}_{f(y)}} \rightarrow \widehat{\mathcal{O}_y}$  between complete local rings is an isomorphism for all points  $y \in Y$ . An étale morphism  $f$  is *Galois* if the induced injection on quotient fields  $\mathbb{k}(X) \rightarrow \mathbb{k}(Y)$  is a Galois extension. The Galois group for this extension acts on  $Y$  with  $X$  being the quotient. A *Galois covering* of  $X$  is a finite Galois étale map  $Y \rightarrow X$ . We say  $X$  is *étale simply connected* if it does not admit any non-trivial Galois coverings. Over  $\mathbb{C}$ , if

the topological fundamental group of  $X$  (in the strong topology) is trivial, then the étale fundamental group is trivial [Mil80].

**Corollary 2.2.** *The variety  $\overline{G}$  is étale simply connected. When  $\mathbb{k} = \mathbb{C}$ , the topological fundamental group of  $\overline{G}$  is trivial.*

*Proof.* Recall that  $G$  is an open dense affine subvariety of  $\overline{G}$ . Since we are over an algebraically closed field, the Bruhat decomposition gives an affine cell in  $G$  that is open and dense [Bor91]. So  $\overline{G}$  is birational to affine space, which actually is birational to projective space. Therefore,  $\overline{G}$  is a rational variety. In general a projective, smooth, rational variety over an algebraically closed field is étale simply connected [Kol03]. Thus,  $\overline{G}$  is étale simply connected.

When  $\mathbb{k} = \mathbb{C}$ , the topological fundamental group of  $\overline{G}$  is trivial, because  $\overline{G}$  is a rational variety [Ser59, p. 483, Proposition 1].  $\square$

**Example 2.3.** In the case of  $G = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ , we have  $\overline{G} = \mathbb{P}(M_2(\mathbb{C})) = \mathbb{CP}^3$  where  $M_2(\mathbb{C})$  is the monoid of  $2 \times 2$  complex matrices. Naturally  $\mathrm{PSL}_2(\mathbb{C}) \subset \mathbb{P}(M_2(\mathbb{C}))$  and the action of  $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$  on  $\mathrm{PSL}_2(\mathbb{C})$  defined by  $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$  extends to an action on  $\mathbb{P}(M_2(\mathbb{C}))$ . The complement  $D = \mathbb{P}(M_2(\mathbb{C})) \setminus \mathrm{PGL}_2(\mathbb{C})$  is the divisor

$$(\{X \in M_2(\mathbb{C}) \mid \det(X) = 0\} \setminus \{\mathbf{0}\}) / \mathbb{C}^* = (\{(a, b, c, d) \in \mathbb{C}^4 \mid ad = bc\} \setminus \{\mathbf{0}\}) / \mathbb{C}^*$$

which is the image of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  under the Segre Embedding. In this divisor, the locus of  $a \neq 0$  is an affine open  $\mathbb{C}^2$ , and when  $a = 0$  we have two copies of  $\mathbb{CP}^1$  intersecting at the point  $[(0, 0, 0, 1)]$ .

### 3. WONDERFUL COMPACTIFICATION OF CHARACTER VARIETIES

In this section, given a finitely generated group  $\Gamma$  and a semisimple algebraic group  $G$  of adjoint type we construct a compactification of the  $G$ -character variety of  $\Gamma$ . There is no assumption on the characteristic of the base field  $\mathbb{k}$ .

We begin with the case of free group. Let  $\Gamma = F_r$  be the free group of rank  $r$ . Then the evaluation mapping gives a bijection  $\mathrm{Hom}(F_r, G) \cong G^r$ . Therefore, as the adjoint action of  $G$  on  $G$  extends to  $\overline{G}$ , the diagonal adjoint action of  $G$  on  $G^r$  also extends to the product  $\overline{G}^r$ ; the action of  $g \in G$  sends any  $(x_1, \dots, x_r) \in \overline{G}^r$  to  $(gx_1g^{-1}, \dots, gx_rg^{-1})$ . Thus,  $\mathrm{Hom}(F_r, G)$  is an affine Zariski open  $G$ -invariant subset of the  $G$ -variety  $\overline{G}^r$ . Then the GIT quotient  $\overline{G}^r // G$ , which is a projective variety, is a compactification of  $\mathfrak{X}_{F_r}(G)$ .

Now let  $\Gamma$  be a finitely generated group, say with  $r$  generators. Fixing  $r$  generators, there is a surjection  $\varphi : F_r \rightarrow \Gamma$  that induces an inclusion  $\varphi_\# : \mathfrak{X}_\Gamma(G) \hookrightarrow \mathfrak{X}_{F_r}(G)$ .

We define the *wonderful compactification* of  $\mathfrak{X}_\Gamma(G)$  to be the closure of  $\mathfrak{X}_\Gamma(G)$  in  $\overline{G}^r // G$  with respect to the above inclusion  $\varphi_\#$ . This wonderful compactification will be denoted by  $\overline{\mathfrak{X}_\Gamma(G)}$ . We note that although up to isomorphism  $\mathfrak{X}_\Gamma(G)$  does not depend on  $\varphi_\#$ , the compactification does (see [Mar11]).

**Theorem 3.1.** *The wonderful compactification  $\overline{\mathfrak{X}_{F_r}(G)}$  is normal and étale simply connected. When  $\mathbb{k} = \mathbb{C}$  it is topologically simply connected.*

*Proof.* Since the GIT quotient of a smooth variety is normal, and  $\overline{G'}$  is smooth, it follows that  $\overline{\mathfrak{X}_{F_r}(G)} \cong \overline{G'} // G$  is normal.

The quotient map  $\overline{G'} \rightarrow \overline{G'} // G$  induces an isomorphism of étale fundamental groups (and topological fundamental groups when  $\mathbb{k} = \mathbb{C}$ ) by [BHP15, Theorem 1]. From Corollary 2.2 we know that  $\overline{G}$  is étale simply connected and therefore the product  $\overline{G'}$  is also étale simply connected. Consequently,  $\overline{\mathfrak{X}_{F_r}(G)} \cong \overline{G'} // G$  is étale simply connected.

If  $\mathbb{k} = \mathbb{C}$ , then  $\overline{G'}$  is topologically simply connected by Corollary 2.2. Hence  $\overline{\mathfrak{X}_{F_r}(G)}$  is topologically simply connected when  $\mathbb{k} = \mathbb{C}$ .  $\square$

**Example 3.2.** Now consider  $\mathfrak{X}_{F_2}(\mathrm{PSL}_2(\mathbb{C})) \cong \mathfrak{X}_{F_2}(\mathrm{SL}_2(\mathbb{C})) / ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}))$ . It is isomorphic to  $\mathbb{C}^3 / ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}))$ , where

$$\mathbb{C}^3 = \{(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \mid A, B \in \mathrm{SL}_2(\mathbb{C})\}.$$

Given Example 2.3, the compactification  $\overline{\mathfrak{X}_{F_2}(\mathrm{PSL}_2(\mathbb{C}))}$  is

$$((M_2(\mathbb{C}) - \{0\}) / \mathbb{C}^* \times (M_2(\mathbb{C}) - \{0\}) / \mathbb{C}^*) // \mathrm{PSL}_2(\mathbb{C}) \cong (\mathbb{C}P^3)^2 // \mathrm{PSL}_2(\mathbb{C}).$$

**Remark 3.3.** In [FL13, Theorem 3.4] it is shown that to each connected quiver  $Q$  and connected reductive algebraic group over  $\mathbb{C}$ , there is an algebraic variety  $\mathcal{M}_Q(G)$  isomorphic to  $\mathfrak{X}_{F_r}(G)$ , where  $r$  is the first Betti number of  $Q$ . In [Man15, Theorem 1.1] it is shown, in the case where  $G$  is simple and simply connected, that each such  $\mathcal{M}_Q(G)$  determines a generally distinct compactification of  $\mathfrak{X}_{F_r}(G)$ . When  $Q$  has exactly one vertex the compactification in [Man15] reduces to the GIT quotient of a product of compactifications of  $G$ , similar to the construction considered here for  $\Gamma = F_r$ . Now the compactification of the group  $G$  considered in [Man15] comes from its so-called Rees algebra. As shown in [KM16, Example 8.12], this compactification of  $G$  coincides with the wonderful compactification of  $G$ . Therefore, our construction is a special case of the construction in [Man15] in the overlapping situation when  $\Gamma$  is free, and  $G$  is a simple, simply connected, complex algebraic group of adjoint type (exactly if  $G$  is one of  $G_2$ ,  $F_4$ , or  $E_8$ ; see [Hu13] for example).

#### 4. SIMPLY CONNECTED COMPACTIFICATIONS OVER $\mathbb{C}$

In this section we work over  $\mathbb{C}$ , and argue that in some cases we can normalize the wonderful compactification of  $\mathfrak{X}_\Gamma(G)$  and obtain simply connected compactifications of character varieties.

We need the following standard result; see [ADH16] and the references therein.

**Proposition 4.1.** *If  $Z$  is a normal projective variety, and  $A \subsetneq Z$  is a closed subvariety, then the natural homomorphism  $\pi_1(Z \setminus A) \rightarrow \pi_1(Z)$  is surjective.*

**Corollary 4.2.** *Let  $G$  be a semisimple adjoint type algebraic group over  $\mathbb{C}$ , and let  $\Gamma$  be either a finitely generated free or free abelian group of rank  $r$ , or the fundamental group of a closed, orientable surface. If  $\overline{\mathfrak{X}_\Gamma(G)}$  is a normal compactification of  $\mathfrak{X}_\Gamma(G)$ , then  $\overline{\mathfrak{X}_\Gamma(G)}$  is simply connected. Consequently,  $\overline{\mathfrak{X}_\Gamma(G)}$  is also étale simply connected.*

*Proof.* For the allowed  $G$  and  $\Gamma$ , it is shown in [BL15, BLR15] that  $\pi_1(\mathfrak{X}_\Gamma(G)) = 1$ . The result now follows from Proposition 4.1.  $\square$

The following two lemmas are standard.

**Lemma 4.3.** *If  $A \subset Z$  is a nonempty Zariski open normal subset of an irreducible projective variety  $Z$ , then the normalization  $\tilde{Z}$  of  $Z$  contains an open subset isomorphic to  $A$ . In particular,  $\tilde{Z}$  is still a compactification of  $A$ .*

**Lemma 4.4.** *Let  $X$  and  $Y$  be normal varieties over an algebraically closed field  $\mathbb{k}$ . Then  $X \times Y$  is also normal.*

With the above lemmas and corollary in mind, we define the normalized wonderful compactification of a normal character variety  $\mathfrak{X}_\Gamma(G)$  to be the normalization of  $\overline{\mathfrak{X}_\Gamma(G)}$ .

**Proposition 4.5.** *Let  $\mathfrak{X}_\Gamma^0(G)$  denote the component of  $\mathfrak{X}_\Gamma(G)$  that contains the trivial representation. In the following cases, the normalized wonderful compactification of  $\mathfrak{X}_\Gamma^0(G)$  is a simply connected compactification of  $\mathfrak{X}_\Gamma^0(G)$  :*

- (1)  $\Gamma = \mathbb{Z}^r$  and  $G$  is any semisimple algebraic adjoint group with no exceptional factors ;
- (2)  $\Gamma = \pi_1 \Sigma$ , with  $\Sigma$  a closed orientable surface, and  $G = \mathrm{PGL}_n$ .

*Proof.* We will show that in both these cases, the character variety  $\mathfrak{X}_\Gamma(G)$  is normal. The result will then follow from Corollary 4.2 and Lemma 4.3.

When  $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n$ , or  $\mathrm{Sp}_n$ , Sikora has shown that  $\mathfrak{X}_{\mathbb{Z}^r}^0(G)$  is normal [Sik14, Theorem 2.1]. Now since the left action of the center of  $G$ , denoted  $Z(G)$ , commutes with the conjugation action of  $G$  on  $\mathrm{Hom}(\mathbb{Z}^r, G)$ , we conclude  $\mathfrak{X}_{\mathbb{Z}^r}(G/Z(G)) \cong \mathfrak{X}_{\mathbb{Z}^r}(G)/Z(G)^r$ . In view of this, since normality is preserved under GIT quotients  $\mathfrak{X}_{\mathbb{Z}^r}(G)$  is likewise normal for  $G = \mathrm{PSL}_n \cong \mathrm{PGL}_n, \mathrm{PSO}_n$ , or  $\mathrm{PSp}_n$ .

Now let  $G$  be a semisimple algebraic adjoint group with no exceptional factors. Then  $G \cong G_1 \times \cdots \times G_n$ , where each  $G_i$  is isomorphic to a simple algebraic adjoint group of type  $A_n, B_n, C_n, D_n$ . By Lemma 4.4 and the previous paragraph  $\mathfrak{X}_{\mathbb{Z}^r}(G_1 \times \cdots \times G_n) \cong \mathfrak{X}_{\mathbb{Z}^r}(G_1) \times \cdots \times \mathfrak{X}_{\mathbb{Z}^r}(G_n)$  is normal.

In the second case, it is a result of Simpson (see [Sim94a, Sim94b]) that  $\mathrm{Hom}(\pi_1 \Sigma, \mathrm{GL}_n)$  is a normal variety. The group  $\mathcal{Z} = \mathrm{Hom}(\pi_1 \Sigma, Z(\mathrm{GL}_n))$ , which is isomorphic to  $\mathbb{G}_m^{b_1(\Sigma)}$ , acts on  $\mathrm{Hom}(\pi_1 \Sigma, \mathrm{GL}_n)$  by left multiplication, and we have

$$\mathrm{Hom}(\pi_1 \Sigma, \mathrm{GL}_n) // \mathcal{Z} \cong \mathrm{Hom}^0(\pi_1 \Sigma, \mathrm{PGL}_n),$$

where the right-hand side denotes the identity component. Since the GIT quotient of a normal variety is normal, we find that  $\mathrm{Hom}^0(\pi_1 \Sigma, \mathrm{PGL}_n)$ , and consequently  $\mathfrak{X}_{\pi_1 \Sigma}^0(\mathrm{PGL}_n)$ , are normal.  $\square$

In [BLR15] we conjecture that for certain groups  $\Gamma$  whose abelianization is free abelian (which we call *exponent canceling groups*), that  $\mathfrak{X}_\Gamma^0(G)$  is simply connected (see [BLR15, Conjecture 2.7]). We also expect that  $\mathfrak{X}_\Gamma(G)$  is normal in these cases. Consequently, we now make:

**Conjecture 4.6.** *The normalized wonderful compactification of  $\mathfrak{X}_\Gamma^0(G)$  is a simply connected compactification of  $\mathfrak{X}_\Gamma^0(G)$  for all exponent canceling  $\Gamma$  and connected adjoint semisimple algebraic groups over  $\mathbb{C}$ .*

## 5. BOUNDARY DIVISORS

In this section we continue to work over  $\mathbb{C}$ . Given a complex projective variety  $X$  with a distinguished dense open affine subvariety  $A \subset X$ , we will use the term *boundary divisor* to refer to hypersurfaces of  $X$  (that is, irreducible codimension 1 subvarieties) contained in  $X \setminus A$ . By Theorem 2.1, the complement  $\overline{G} \setminus G$  is a union of  $r = \text{rank}(G)$  smooth boundary divisors, and each of these divisors is the closure of a  $G \times G$ -orbit.

Now let  $D_i$  be a boundary divisor of  $\overline{G}$ . Then there exist

$$\mathbf{m}_{I_1}, \dots, \mathbf{m}_{I_{m_i}} \in \text{Gr}(n, \mathfrak{g} \times \mathfrak{g}),$$

where each  $I_j \subset \{1, \dots, r\}$ , so that

$$D_i = \cup_j (G \times G) \cdot \mathbf{m}_{I_j} \cong \cup_j (G \times G) / \text{Stab}(\mathbf{m}_{I_j}).$$

In particular, each boundary divisor is isomorphic to a union of homogeneous spaces, each a quotient by a closed subgroup (since stabilizers of algebraic group actions are always algebraic subgroups).

**Lemma 5.1.** *Let  $V$  be an affine  $G$ -variety and  $W$  a compactification of  $V$  on which the  $G$ -action extends. Assume that each boundary divisor of  $W$  is saturated with respect to the GIT quotient map  $W \rightarrow W//G$ . Then the boundary divisors of  $W//G$  (with respect to the subvariety  $V//G$ ) are simply the components of  $(W \setminus V)//G$ .*

*Proof.* As the  $G$ -action extends to  $W$ , we see that  $V$  is a  $G$ -stable affine open subset of  $W$ , and the boundary divisors in  $W \setminus V$  are unions of  $G$ -orbits. The usual gluing construction for the GIT quotient (see [Dol03]) shows that  $V//G$  is an affine open in  $W//G$ . Since the boundary divisors in  $W \setminus V$  are saturated,  $W \setminus V$  is itself saturated, so we find that  $(W//G) \setminus (V//G)$  is exactly  $(\cup_i D_i)//G$  where the  $D_i$ 's are the boundary divisors in  $W \setminus V$ .  $\square$

In [BFL14] parabolic character varieties of free groups are defined and studied. We recall their definition. Let  $G$  be a complex reductive group, and let  $G_1, \dots, G_m$  be closed subgroups. Then  $G$  acts on the *mixed product*

$$G^n \times \prod_{1 \leq j \leq m} G/G_j$$

by

$$g \cdot (h_1, \dots, h_n, g_1 G_1, \dots, g_m G_m) = (gh_1 g^{-1}, \dots, gh_n g^{-1}, gg_1 G_1, \dots, gg_m G_m).$$

The quotient  $(G^n \times \prod_{1 \leq j \leq m} G/G_j)//G$  is the parabolic character variety of the free group of rank  $n$  with parabolic data  $\{G/G_j\}_{j=1}^m$ . We note that when the  $G_i$ 's are reductive, as assumed in [BFL14], the homogeneous spaces  $G/G_i$  are affine, and when the  $G_i$ 's are parabolic, the homogeneous spaces  $G/G_i$  are projective. In general, they are quasi-projective [Bor91, Theorem 6.8], and the GIT quotient defining the parabolic character varieties of free groups is taken in that setting.

**Theorem 5.2.** *The boundary divisors in  $\overline{G}^r//G$  are unions of parabolic character varieties of free groups.*

*Proof.* As noted above the boundary divisors in  $\overline{G}$  are unions of homogeneous spaces of  $G \times G$ , and by Theorem 2.1 each boundary divisor is the closure of a single  $G \times G$ -orbit. Therefore, in  $\overline{G}$  the boundary divisors are unions of mixed products of  $G \times G$ -homogeneous spaces, and in fact are saturated with respect to the GIT quotient map for the conjugation action. The action of conjugation on an orbit corresponds, under the isomorphism between the orbit and the corresponding homogeneous space, to the left action on the homogeneous space. Thus, by Lemma 5.1 and the definition of parabolic character variety of free groups, the result follows.  $\square$

**Example 5.3.** In Example 2.3 we see that the sole boundary divisor of the wonderful compactification of  $\mathrm{PSL}_2(\mathbb{C})$  is isomorphic to  $\mathbb{CP}^1 \times \mathbb{CP}^1$ , a product of homogeneous spaces. Therefore, the boundary divisors in Example 3.2, namely in  $\overline{\mathfrak{X}_{F_2}(\mathrm{PSL}_2(\mathbb{C}))}$ , arise as the GIT quotient of the diagonal left multiplication action of  $\mathrm{PSL}_2(\mathbb{C})$  on products of  $\mathbb{CP}^1 \times \mathbb{CP}^1$ ; this is an example of a parabolic character variety as it is a left diagonal quotient of a product of homogeneous spaces.

## 6. POISSON STRUCTURE

Recall that a Poisson algebra is a Lie algebra in which the Lie bracket is also a derivation in each variable. We call a quasi-projective variety  $X$  over  $\mathbb{C}$  a Poisson variety if the sheaf of smooth functions on  $X^{sm}$  (where  $X^{sm}$  is the smooth locus of  $X$ ) is a sheaf of Poisson algebras; in other words, if each on each open subset  $U \subset X^{sm}$  (in the analytic topology) the algebra of holomorphic functions  $C^\infty(U, \mathbb{C})$  is equipped with a Poisson algebra structure (so that restriction respects the Poisson brackets). Recall that, in general, complex Poisson manifolds admit  $(2, 0)$ -symplectic foliations.

A Poisson bracket on  $C^\infty(X^{sm}, \mathbb{C})$  (the algebra of *holomorphic* functions) corresponds to an exterior bivector field  $\mathfrak{a} \in \Lambda^2(T^{1,0}X)$  whose restriction to symplectic leaves is given by the symplectic form:  $\{f, g\} = \omega(H_g, H_f)$ , where  $H_f$  is the contraction of  $\mathfrak{a}$  and  $df$  ( $H_f$  is called the Hamiltonian vector field associated to  $f$ ). If  $f, g \in C^\infty(X^{sm}, \mathbb{C})$ , then, with respect to interior multiplication,  $\{f, g\} = \mathfrak{a} \cdot df \otimes dg$ . In local (complex) coordinates  $(z_1, \dots, z_k)$  it takes the form

$$\mathfrak{a} = \sum_{i,j} \mathfrak{a}_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

and so

$$\begin{aligned} \{f, g\} &= \sum_{i,j} \left( \mathfrak{a}_{i,j} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \right) \cdot \left( \frac{\partial f}{\partial z_i} dz_i \otimes \frac{\partial g}{\partial z_j} dz_j \right) \\ &= \sum_{i,j} \mathfrak{a}_{i,j} \left( \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial z_i} \right). \end{aligned} \tag{6.1}$$

For the rest of the section,  $G$  will denote a adjoint type semisimple algebraic group over  $\mathbb{C}$ . In [EL01, EL06] it is shown that  $\overline{G}$  has a Poisson structure which we denote by  $\{ , \}_{\overline{G}}$ . Moreover, they show that it restricts to each  $G \times G$ -orbit and hence to each boundary divisor. We will review the construction of this Poisson structure in the course of proving the next result.

For a  $G \times G$ -space  $X$ , we will refer to the  $G$ -action  $g \cdot x = (g, g) \cdot x$  as the *adjoint action* of  $G$  on  $X$ . Functions on  $X$  satisfying  $f(g \cdot x) = f(x)$  will be called *Ad-invariant*.



**Lemma 6.1.** *The Poisson structure on  $\overline{G}$  (and on each smooth  $G \times G$ -invariant subvariety of  $\overline{G}$ ) preserves the subalgebra of Ad-invariant functions. In other words, if  $f_1$  and  $f_2$  are Ad-invariant, then so is  $\{f, g\}_{\overline{G}}$ . Equivalently, the bivector  $\mathfrak{a}$  defining this Poisson bracket is Ad-invariant.*

*Proof.* We will work in the general context of a complex  $G \times G$ -manifold  $X$  (since  $\overline{G}$  is a smooth  $G \times G$ -space) where  $G$  is a Lie group equipped with an Ad-invariant non-degenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . In this context, Evens and Lu (see [EL01, EL06]) describe a bracket on  $C^\infty(X, \mathbb{C})$  that gives rise to the desired Poisson structures. We now describe this bracket in order to examine its interaction with the adjoint action of  $G$ .

Let  $\mathfrak{g} = \text{Lie}(G)$  and let  $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ . Note that the adjoint action of  $G$  on  $\mathfrak{g}$  makes  $\mathfrak{d}$  into a  $G \times G$ -space. Define  $R^* \in (\wedge^2 \mathfrak{d})^*$  to be the linear functional associated to the alternating bilinear form

$$\mathfrak{d} \times \mathfrak{d} \longrightarrow \mathbb{C}$$

given by

$$((y_1, x_1), (y_2, x_2)) \longmapsto \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle.$$

The action of  $G \times G$  on  $\mathfrak{d}$  extends to an action on  $\wedge^2 \mathfrak{d}$ , and with respect to this action,  $R^*$  is an Ad-invariant linear functional on  $\wedge^2 \mathfrak{d}$ . Explicitly, Ad-invariance of the bilinear form yields

$$R^*((\text{Ad}(g)y_1, \text{Ad}(g)x_1) \wedge (\text{Ad}(g)y_2, \text{Ad}(g)x_2)) = R^*((y_1, x_1) \wedge (y_2, x_2)),$$

where  $\text{Ad}(g)z$  denotes the adjoint action of  $g \in G$  on  $z \in \mathfrak{g}$ .

The Ad-invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  induces Ad-invariant forms on  $\mathfrak{d}$  and on  $\wedge^2 \mathfrak{d}$ . The latter bilinear form (which we still denote simply by  $\langle \cdot, \cdot \rangle$ ) induces an isomorphism

$$\wedge^2 \mathfrak{d} \xrightarrow{\cong} (\wedge^2 \mathfrak{d})^*.$$

Letting  $R$  denote the preimage of  $R^*$  under this isomorphism, we see that  $R$  itself is invariant under the adjoint action of  $G$ : fixing  $g \in G$ , we have

$$\langle g \cdot R, \eta \rangle = \langle R, g^{-1} \cdot \eta \rangle = R^*(g^{-1} \cdot \eta) = R^*(\eta) = \langle R, \eta \rangle$$

for each  $\eta \in \wedge^2 \mathfrak{d}$ , so we find that  $g \cdot R = R$ .

Associated to the  $G \times G$ -space  $X$  we have a natural map

$$\kappa: \mathfrak{d} \longrightarrow \chi^1(X),$$

where  $\chi^1(X)$  denotes the space of (holomorphic) vector fields on  $X$ . This map is defined by representing an element in  $d \in \mathfrak{d}$  as a one-parameter subgroup  $(g_t, h_t)$  in  $G \times G$  and letting  $\kappa(d)_x$  (the value of the vector field  $\kappa(d)$  at the point  $x \in X$ ) be the vector represented by the curve  $(g_t, h_t) \cdot x$ . It follows from the definition that the map  $\kappa$  has the following equivariance property: for each  $k \in G \times G$ ,  $d \in \mathfrak{d}$ , and  $x \in X$ , we have

$$\kappa(k \cdot d)_{k \cdot x} = k \cdot \kappa(d)_x, \tag{6.2}$$

where on the right-hand side  $k \cdot -$  denotes the derivative of the map  $X \rightarrow X$ , given by  $x \mapsto k \cdot x$ .

The map  $\kappa$  induces a map

$$\kappa: \wedge^2 \mathfrak{d} \longrightarrow \wedge^2 \chi^1(X)$$

that satisfies the corresponding version of (6.2). This implies that  $\kappa$  carries Ad-invariant forms to Ad-equivariant vector fields. In particular, Ad-invariance of  $R$  implies that

$$\kappa(R)_{(g,g) \cdot x} = \kappa((g, g) \cdot R)_{(g,g) \cdot x} = (g, g) \cdot \kappa(R)_x$$

for each  $g \in G$ ,  $x \in X$ .

Let  $\Pi = \frac{1}{2}\kappa(R)$ . The bracket on  $C^\infty(X, \mathbb{C})$  is defined as follows. Associated to functions  $f_1, f_2 \in C^\infty(X, \mathbb{C})$ , we have holomorphic 1-forms  $df_1, df_2 \in \chi_1(X)$ , the space of sections of the holomorphic cotangent bundle of  $X$ . There is a natural pairing

$$\langle \cdot, \cdot \rangle: \wedge^2(\chi^1(X)) \times \wedge^2(\chi_1(X)) \longrightarrow C^\infty(X, \mathbb{C}) \quad (6.3)$$

induced by the map

$$\chi^1(X) \times \chi^1(X) \times \chi_1(X) \times \chi_1(X) \longrightarrow C^\infty(X, \mathbb{C})$$

defined by

$$(\alpha, \beta, V, W) \longmapsto \alpha(V)\beta(W) - \alpha(W)\beta(V), \quad (6.4)$$

where  $\alpha(V)$  denotes the result of evaluating a section of the cotangent bundle on a section of the tangent bundle. Note that (6.4) is alternating in the first two variables and in the last two variables, so it induces the desired map (6.3). We now define

$$\{f_1, f_2\} = \langle df_1 \wedge df_2, \Pi \rangle.$$

To complete the proof, we need to show that when  $f_1$  and  $f_2$  are Ad-invariant, so is  $\{f_1, f_2\}$ . In this situation, we have:

$$\begin{aligned} \{f_1, f_2\}((g, g) \cdot x) &= \langle (df_1)_{(g,g) \cdot x} \wedge (df_2)_{(g,g) \cdot x}, \Pi_{(g,g) \cdot x} \rangle \\ &= \langle (df_1)_{(g,g) \cdot x} \wedge (df_2)_{(g,g) \cdot x}, (g, g) \cdot \Pi_x \rangle \\ &= \langle (df_1)_x \wedge (df_2)_x, \Pi_x \rangle \\ &= \{f_1, f_2\}(x), \end{aligned}$$

since  $f_1((g, g) \cdot -) = f_1(-)$  implies  $(df_1)_{(g,g) \cdot x}((g, g) \cdot v) = (df_1)_x(v)$  and similarly for  $f_2$ .  $\square$

**Lemma 6.2.** *There exists a Poisson structure on the smooth projective variety  $\overline{G}^r$  whose bivector  $\mathbf{a}$  is invariant under conjugation by  $G$ .*

*Proof.* Given Poisson manifolds  $X$  and  $Y$  with bivectors  $\mathbf{a}_X$  and  $\mathbf{a}_Y$  respectively,  $X \times Y$  is a Poisson manifold with bivector  $\mathbf{a}_{X \times Y} = \mathbf{a}_X + \mathbf{a}_Y$ . This is the unique Poisson structure on  $X \times Y$  so that the projections  $X \times Y \rightarrow X$  and  $X \times Y \rightarrow Y$  are Poisson maps (see [Vai94]). Thus, existence follows by taking  $\mathbf{a}_{\overline{G}^r} = \sum_{i=1}^r \mathbf{a}_{\overline{G}}$ .

Now we need to prove  $G$ -invariance of the bivector. This follows since the construction of the bivector in [EL01, EL06], which is on  $\text{Gr}(n, \mathfrak{g} \times \mathfrak{g})$  and restricted to the  $G \times G$ -orbits, is Ad-invariant by Lemma 6.1.  $\square$

It is well-known that if  $X$  is a Poisson manifold and a Lie group  $G$  acts on  $X$  through Poisson maps, then the  $G$ -invariant functions on  $X$  form a Poisson algebra (see for instance [DZ05, p. 24]). The following lemma is a version of this statement.

**Lemma 6.3.** *If a complex Poisson variety  $X$  admits the action of a reductive algebraic group  $G$  and the bivector is invariant, then the Poisson bracket on  $C^\infty(X, \mathbb{C})$  restricts to a Poisson structure on the  $G$ -invariant holomorphic functions  $C^\infty(X, \mathbb{C})^G$ . Consequently, the GIT quotient  $X//G$  is a Poisson variety.*

*Proof.* Let the action of  $g \in G$  be denoted by  $\alpha_g$ , so  $g \cdot x = \alpha_g(x)$  for all  $x \in X$ . Also, let the Poisson bracket on  $X$  be denoted by  $\{ , \}_X$ . For  $f, g \in C^\infty(X, \mathbb{C})^G$  we have

$$g \cdot \{f_1, f_2\}_X = \{f_1, f_2\}_X \circ \alpha_{g^{-1}} = \alpha_{g^{-1}}^* \{f_1, f_2\}_X.$$

Since the functional coefficients  $\mathbf{a}_{i,j}$  are  $G$ -invariant, Equation 6.1 implies

$$\alpha_{g^{-1}}^* \{f_1, f_2\}_X = \{\alpha_{g^{-1}}^* f_1, \alpha_{g^{-1}}^* f_2\}_X = \{g \cdot f_1, g \cdot f_2\}_X = \{f_1, f_2\}_X.$$

Thus, the restriction of  $\{ , \}_X$  to  $C^\infty(X, \mathbb{C})^G \times C^\infty(X, \mathbb{C})^G$  has image contained in  $C^\infty(X, \mathbb{C})^G$ . All other properties of the Lie bracket, as well as satisfying the Leibniz rule, are inherited from  $\{ , \}_X$  once we know this latter fact. Since the holomorphic functions on an open subset  $U \subset (X//G)^{sm}$  (in the analytic topology) correspond to the  $G$ -invariant holomorphic functions on the preimage of  $U$  in  $X$ , this defines the desired Poisson structure on  $X//G$ .  $\square$

**Theorem 6.4.** *There exists a Poisson structure on the wonderful compactification of a free group character variety over  $\mathbb{C}$ , and also on its boundary divisors.*

*Proof.* The Poisson structure on  $\overline{G}$  gives an  $\text{Ad}$ -invariant Poisson structure on  $\overline{G}^r$  by Lemma 6.2, and descends to a Poisson structure on  $\overline{\mathfrak{X}_{F_r}(G)}$  by Lemma 6.3.

Since each boundary divisor of  $\overline{G}^r$  is a union of products of orbits where each admits a Poisson structure (restriction from that on  $\overline{G}$ ), the same argument as above shows that the Poisson structures on the boundary divisors of  $\overline{G}^r$  descend to the boundary divisors of  $\overline{\mathfrak{X}_{F_r}(G)}$ .  $\square$

Since the boundary divisors of  $\overline{\mathfrak{X}_{F_r}(G)}$  are unions of parabolic free group character varieties, we immediately conclude:

**Corollary 6.5.** *There exists a Poisson structure on those parabolic character varieties of free groups that lie inside the boundary divisors of  $\overline{\mathfrak{X}_{F_r}(G)}$ .*

We call the Poisson structure shown to exist above the *wonderful Poisson structure*.

In [Gol86, Gol84] Goldman showed there is a Poisson structure on  $\text{Hom}(\pi_1 \Sigma_{g,n})//G$  where  $\Sigma_{g,n}$  is an orientable surface of genus  $g$  with  $n$  disjoint boundary components. Moreover, the Casimirs (those functions that Poisson commute) are exactly the invariant functions restricted to the boundary components. See [Law09] also.

**Question 6.6.** *How does Goldman's Poisson structure on  $G^r//G$  relate to the wonderful Poisson structure on  $G^r//G$ ,  $\overline{G}^r//G$ , and its boundary divisors?*

**Remark 6.7.** Given an affine Poisson variety  $V$ , the Poisson bracket  $\{ , \}_V$  is determined by its action on the coordinate ring  $\mathbb{C}[V]$  by the Stone-Weierstrass theorem. Suppose  $V$  has Casimirs  $\{c_1, \dots, c_m\}$ . Then the algebra  $A := \mathbb{C}[V]/(c_1 - \lambda_1, \dots, c_m - \lambda_m)$ , where  $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ , is a Poisson algebra with bracket defined by  $\{f + I, g + I\} = \{f, g\}_V + I$  where  $I$  is the ideal  $(c_1 - \lambda_1, \dots, c_m - \lambda_m)$ . Therefore, the variety  $\text{Spec}(A)$  is an affine Poisson variety.

Now applying Remark 6.7 to the setting of parabolic character varieties of free groups we see that whenever the parabolic data  $\{G/H_i\}$  are isomorphic to  $G$ -conjugation orbits (equivalently  $H_i$ 's are isomorphic to conjugation stabilizers), then the Goldman Poisson

bracket on  $G^r // G$  with some set of its Casimirs fixed (fixing some set of the boundaries up to conjugation is equivalent to fixing some set of the Casimirs) determines a Poisson structure on the parabolic character variety of a free group resulting from fixing some (but not all) the boundaries to conjugation orbits. Therefore, we have a Goldman-type Poisson structure on certain parabolic character varieties of free groups.

**Question 6.8.** *How does this Goldman-type Poisson structure compare to the wonderful Poisson structure from Corollary 6.5?*

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